Lecture 2

1. Recap (tensors - ex. session)
2. EM waves
3. Integral form of M.E.
4. EM potentials
5. Gauge invariance

- Methods for solutions of ME in electrostatics
  1. Green Functions Method
     → complex calculus recap
  2. Image charge method
  3. Expansion in Eigenfunctions
- Magnetostatics

Curl: $\vec{\nabla}\times\vec{A} = \varepsilon_{ijk}\nabla_j A_k e_i$

Laplacian: $\vec{\nabla}\cdot\vec{\nabla}\vartheta \equiv \Delta\vartheta = \partial_i\partial_i\vartheta$

Gauss theorem

$$\int_R \vec{\nabla}\cdot\vec{E}\,dV = \oint_{\text{Boundary}(R)} \vec{E}\cdot d\vec{S}$$

↳ normal vector

Stokes theorem

$$\int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{B}) = \oint_{\text{boundary}(S)} d\vec{\ell}\cdot\vec{B}$$

Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

charge:

$$\vec{\partial}\cdot\vec{J} + \dot{\rho} = 0$$

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2. E.M. waves

Consider ME in vacuum:

$$\partial_t\left[\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

$$\vec{\nabla}\times\left[\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right]$$

$$\vec{\nabla}\cdot\vec{E} = 0.$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \varepsilon^{ijk}\partial_j \varepsilon^{klm}\partial_l E^m$$

$$\varepsilon^{ijk}\varepsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$$

$$\delta^{il}\partial_l \partial_j E^j = 0 \quad (\vec{\nabla}\cdot\vec{E}) = 0 \quad \Rightarrow \text{ we get}$$

$$-\partial_j \partial_j \delta^{im} E = -\Delta \vec{E} \quad \Rightarrow$$

$$-\Delta \vec{E} = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t}$$

$$\partial_t \vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \Delta \vec{E} = 0$$

This is the wave equation. Function

$f(z-ct)$ with $c^2 = \frac{1}{\mu_0 \varepsilon_0}$ is

a solution. $c \equiv$ speed of light!

$\vec{B}$ satisfies the same equation:

$$\Delta B - \mu_0 \varepsilon_0 \ddot{B} = 0$$

History $\quad$ ME ~ 1862 – 1864

Radio ~ 1894

ε_0 is redundant:

$$\rho, J \to \sqrt{\varepsilon_0}\,\rho, \sqrt{\varepsilon_0}\,J$$

$$E, B \to \frac{1}{\sqrt{\varepsilon_0}} E, \frac{1}{\sqrt{\varepsilon_0}} B$$

this rescaling removes ε_0 from ME. The only physical quantity is $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

3. Integral form of Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\int_R \vec{\nabla}\cdot\vec{E}\, dV = \oint \vec{E}\cdot d\vec{S}$$

$$\int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{B}) = \oint d\vec{l}\cdot\vec{B}$$

$$\oint E\cdot dS = \int_V d^3x \frac{\rho}{\varepsilon_0} = \frac{1}{\varepsilon_0} Q$$

$$\oint dl\cdot B = \mu_0 \int d\vec{\sigma}\cdot\vec{J} + \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{\sigma}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int d\vec{\sigma}\cdot\vec{B}$$

$$\oint B\cdot dS = 0$$

4. EM Potentials

There is an equivalent description of CED in terms of EM potentials which is often more convenient

$\vec{B}$ is div.-free

- $\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad \vec{A} = -\frac{\vec{\nabla} \times \vec{B}}{\Delta}$ works

Let's check: $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \cancel{\vec{\nabla}(\vec{\nabla} \cdot \vec{B})}^{\,0} - \Delta \vec{B}$

- $\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = 0 \iff \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$

$\vec{\nabla} \times \vec{E}' = 0 \quad (\vec{E}' = \vec{E} + \dot{\vec{A}})$ $\quad \vec{E}'$ is curl-free

$\vec{E}' = -\vec{\nabla} \cdot \varphi$ $\qquad \varphi = -\frac{\vec{\nabla} \cdot \vec{E}'}{\Delta}$ works

Let's check:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) = \cancel{\vec{\nabla} \times \vec{\nabla} \times \vec{E}}^{\,0} + \Delta \vec{E}$$

Finally:

$$\vec{E} = -\vec{\nabla} \cdot \varphi - \partial_t \vec{A}$$

We can substitute the definitions of potentials into ME to find the equations that potentials satisfy:

$$\vec\nabla\cdot\vec E = \frac{\rho}{\varepsilon_0}$$
$$\vec\nabla\times\vec B = \mu_0\vec J + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}$$
$$\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t}$$
$$\vec\nabla\cdot\vec B = 0$$

MI:
$$-\Delta\varphi - \partial_t\vec\nabla\cdot\vec A = \frac{\rho}{\varepsilon_0}$$

MII:
$$-\Delta\vec A + \vec\nabla(\vec\nabla\cdot\vec A) = \mu_0\vec J - \frac{\partial}{\partial t}\vec\nabla\varphi\,\mu_0\varepsilon_0 - \partial_t^2\vec A\,\mu_0\varepsilon_0$$

$$\frac{1}{c^2}\partial_t^2\vec A - \Delta\vec A = \mu_0\vec J - \vec\nabla\left(\frac{\partial_t\varphi}{c^2} + \vec\nabla\cdot\vec A\right)$$

To bring MI to the similar form:

$$\frac{1}{c^2}\partial_t^2\varphi - \Delta\varphi = \frac{\rho}{\varepsilon_0} + \partial_t\left(\frac{\partial_t\varphi}{c^2} + \vec\nabla\cdot\vec A\right)$$

M III and M IV are automatic due to the relation of $\vec{E}, \vec{B}$ to $\varphi, \vec{A}$

5. Gauge invariance

$\varphi, \vec{A}$ uniquely determine $\vec{E}$ and $\vec{B}$.

But different φ and $\vec{A}$ can correspond to the same $\vec{E}$ and $\vec{B}$.

Indeed:

$$\vec{E} = -\vec{\nabla}\varphi - \partial_t \vec{A} \qquad \left\{ \begin{array}{l} \varphi \to \varphi + \dot{\alpha} \\ \vec{A} \to \vec{A} - \vec{\nabla}\alpha \end{array} \right.$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Leaves $\vec{E}, \vec{B}$ invariant. This ambiguity is called gauge invariance. It has profound consequences. Only $\vec{E}$ and $\vec{B}$ are observable, hence we can use gauge transformations to simplify the potentials.

We will use it to enforce an extra equation that they satisfy (Lorenz gauge condition)

$$\frac{1}{c^2}\partial_t \varphi + \vec{\nabla}\cdot\vec{A} = 0$$

Let us check that it is possible:

$$\frac{1}{c^2}\partial_t \varphi + \frac{1}{c^2}\partial_t^2 \alpha - \Delta\alpha + \vec{\nabla}\cdot\vec{A} = 0$$

$$\Box\alpha + \frac{1}{c^2}\partial_t \varphi + \vec{\nabla}\cdot\vec{A} = 0$$

$\hookrightarrow$ possible to find α

Now ME simplify a lot:

$$\boxed{\begin{aligned} \frac{1}{c^2}\partial_t^2 \vec{A} - \Delta\vec{A} &= \mu_0 \vec{J} \\ \frac{1}{c^2}\partial_t^2 \varphi - \Delta\varphi &= \frac{\rho}{\varepsilon_0} \end{aligned}}$$

$$\frac{1}{c^2}\partial_t^2 - \Delta \equiv \Box$$

Note similarity for $\vec{A}$ and φ.

Below we will use this form of ME.

Given $\vec{A}$ and φ one finds

Methods of solving ME in Electrostatics

$$\vec{A} = \vec{B} = 0 \;, \qquad \vec{E} = -\vec{\nabla}\varphi \;, \text{ with}$$

$$\Delta\Phi = -\frac{\rho}{\varepsilon_0}$$

If all charges are known, and

$\varphi \xrightarrow[\vec{x}\to\infty]{} 0$ then

$$\Phi = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(x')}{|x-x'|}$$

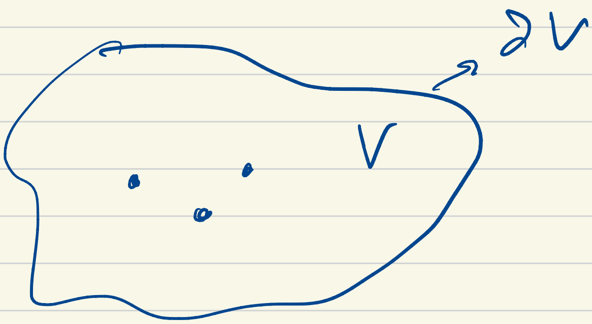
This is "sum" over point-like charges

To find φ in more complicated situations, we need more advanced tools.

1. Greens Functions Method

[Sadcson]

Suppose we are given a set of charges inside some region, and boundary conditions on Φ.



[usually $\Phi = 0$ or $\partial_n \Phi = 0$]

To find Φ inside V we would like to first solve for the greens function with given B.C.:

$$-\Delta_x G(x, x') = \delta^3(x - x')$$

If $\Delta_x F(x, x') = 0$ then

$G + F$ is also Green's function, but with different B.C.

For Dirichlet and Neumann B.C. the G.F. is unique. Let us prove it

Neumann: $\vec{n}\cdot\vec{E} = -\vec{n}\cdot\vec{\nabla}\varphi\Big|_{\partial V} = f(x)$

Dirichlet: $\varphi\Big|_{\partial V} = g(x)$

As we will see below, useful G.F. are those with homogeneous B.C. ($f=0$ or $g=0$).

Let's start with proving a master formula:

$$\varphi(x) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G(x',x)\,\rho(x') +$$

$$+\int_{\partial V} d\vec{\sigma}'\cdot\left(G(x',x)\vec{\nabla}_{x'}\varphi(x') - \varphi(x')\vec{\nabla}_{x'}G(x',x)\right)$$

(*)

to prove let's define the quantity F:

$$F \equiv \int d^3x' \, \vec{\nabla}_{x'} \left(G(x', x) \vec{\nabla}_{x'} \varphi(x') - \varphi(x') \vec{\nabla}_{x'} G(x', x) \right) =$$

$$= -\int G(x', x) \frac{\rho(x')}{\varepsilon} + \int d^3x' \, \delta^3(\vec{x} - \vec{x}') \cdot \varphi(x') =$$

used $\Delta \varphi = -\frac{\rho}{\varepsilon_0}$ used $\Delta_{x'} G(x', x) = \delta^3(x' - x)$

$$= \varphi(x) - \int G(x', x) \frac{\rho(x')}{\varepsilon_0} \quad (1)$$

On the other hand, Gauss theorem applied to F gives

$$F = \int_{\partial V} d\vec{\sigma}' \left(G(x', x) \vec{\nabla}_{x'} \varphi(x') - \varphi(x') \vec{\nabla}_{x'} G(x', x) \right) \quad (2)$$

Combining (1) and (2) gives (*)

(*) is true for any boundary conditions on G or φ.
Depending on the problem it is convenient to choose G_N or G_D

$$\varphi(x)\Big|_{\partial V} = g(x) \rightarrow \text{chose } G_D: \ G_D\Big|_{\partial V} = 0$$

$$\varphi(x) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G(x',x)\,\rho(x') - \int_{\partial V} d\vec{\sigma}\, g(x')\,\vec{\nabla}_{x'} G_D(x',x)$$

If, instead,

$$-\vec{n}\cdot\vec{\nabla}\varphi\Big|_{\partial V} = f(x) \rightarrow \text{choose } G_N: \vec{n}\cdot\vec{\nabla}G_N\Big|_{\partial V} = 0$$

$$\varphi(x) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G(x',x)\,\rho(x') +$$

$$+\int_{\partial V} d\vec{\sigma}'\cdot G_N(x',x)\,\vec{\nabla}\varphi(x')$$

In what follows we will use Uniqueness of the solution to N and D problems.

Suppose there are two solutions φ_1 and φ_2

$$\Delta\varphi_i = -\frac{\rho}{\varepsilon_0}, \quad \varphi_i\Big|_{\partial V} = g, \text{ or } \partial_n\varphi_i\Big|_{\partial V} = g$$

$i=1,2$

Consider $U = \varphi_1 - \varphi_2$, and integrate

$$\Delta U = 0$$

$$\int_V \vec{\nabla}(U\vec{\nabla}U) = \int_V (\vec{\nabla}U)^2 \Rightarrow \vec{\nabla}U = 0$$

$\Downarrow$

$\varphi_1 - \varphi_2 = \text{const.}$

$\parallel \to$ Gauss

$$\int_{\partial V} d\vec{\sigma}(U\vec{\nabla}U) = 0 \quad \text{(either } U \text{ or } \vec{\nabla}U \text{ vanishes)}$$

It follows that G_D and G_N are also unique (up to adding a const to G_N)